J. Fluid Mech. (1971), *vol.* 45, *part* 3, *pp.* 585-600
Printed in Great Britain

Asymptotic solution for plume at very large and small Prandtl numbers

By H. K. KUIKEN† AND ZEËV ROTEM
Department of Mechanical Engineering, University of British Columbia, Vancouver, Canada

(Received 9 July 1969 and in revised form 6 April 1970)

The investigation of laminar free convective plumes in an otherwise stationary environment has formed the basis of numerous investigations, initiated by Zeldovich (1937). For the non-rotating environment alone the authors have been able to locate twenty-nine papers: many of these repeat work previously undertaken. There are, however, two cases of some technological significance which have so far not been considered: (i) the plume in an otherwise quiescent environment for a fluid of very large Prandtl number, of importance in the heating of reservoirs of viscous fluid such as fuel oil; and (ii) the case of vanishingly small Prandtl number, of application to liquid metal-cooled nuclear reactors. Both of these cases have some theoretical interest, as will be shown. Their analysis leads to singular asymptotic perturbations and hence to matched-expansions techniques.

1. Introduction

The behaviour of both laminar and turbulent free convective plumes in an unconfined environment is well known. However, with the sole exception of a paper by Spalding & Cruddace (1961), the asymptotic cases for extreme values of the Prandtl number σ have not been analysed. Those authors obtained the first term of an 'outer' solution for the fluid of $\sigma \gg 1$ on a semi-intuitive basis. The reason for this seems to be the singular behaviour of a plume at very large and at very small σ. Whereas the derivation of similar flows over vertical or horizontal plates is comparatively simple (Kuiken 1968*b*, 1969; Rotem & Claassen 1969*a*), this is not the case for the thermal plume: here the velocity does not vanish in the plane of symmetry of the jet. For the configuration with a solid boundary, the 'no-slip' condition assures the disappearance of the velocity, while in the present case it is rather the shear-stress which vanishes in the centre plane. In the method employed in the analysis presented, two asymptotic expressions will be derived for each case by choosing different width scales: the one valid near the axis, the other for the rest of the plume. Matching will be accomplished in the well-known way (Cole 1968). Uniformly valid approximations will also be given and compared to data calculated numerically. We shall show that the nature of the solutions obtained differs considerably from that

† Present address: Philips Research Laboratories, N.V. Philips' Gloeilampenfabrieker, Eindhoven, The Netherlands.

known for flow over a solid boundary. As the way in which the asymptotic derivations will be obtained is not immediately obvious, a model problem will be investigated which will show the essential features.

2. The model example

Let an infinite vertical plane heat source be given forming an imaginary surface immersed in a fluid initially at rest. Let x denote time and T_∞ the initial temperature of the surrounding fluid, y the co-ordinate normal to the plane. For $x = 0$, let the temperature of the plane increase to some large initial value and vary thereafter as $x^{-\frac{1}{2}}$. Then, for times larger than 0, the fluid will be set in motion due to buoyancy forces depending upon x and y alone. The equations of motion and energy-transport reduce to

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial^2 u}{\partial y^2} + \theta, \\ \sigma \frac{\partial \theta}{\partial x} &= \frac{\partial^2 \theta}{\partial y^2}, \end{aligned}\right\} \quad (x > 0). \tag{1}$$

Here σ is a parameter, later to be identified with the Prandtl number. The boundary conditions imposed are

$$\left.\begin{aligned} \partial u/\partial y|_{y=0} = 0, \quad \theta \propto x^{-\frac{1}{2}} \quad (x > 0), \\ u = \theta = 0, \quad \text{as} \quad y \to \infty. \end{aligned}\right\} \tag{2}$$

The solutions found by elementary means are as follows:

$$\left.\begin{aligned} u(x,y) &= \frac{2}{\sigma - 1}(\pi x)^{\frac{1}{2}}.\left[\sigma^{\frac{1}{2}}\,\text{ierfc}\left(\frac{y}{2x^{\frac{1}{2}}}\right) - \text{ierfc}\left\{\frac{y}{2}\left(\frac{\sigma}{x}\right)^{\frac{1}{2}}\right\}\right], \\ \theta(x,y) &= x^{-\frac{1}{2}}\exp\left[-\left(\frac{y}{2}\right)^2\frac{\sigma}{x}\right]. \end{aligned}\right\} \tag{3}$$

For large values of σ the double-layer character of this model problem becomes evident: the temperature θ will tend to the value of zero in a region in which the variable $\eta = (y/2)(\sigma/x)^{\frac{1}{2}}$ exceeds order unity, while the velocity will tend towards zero in a different region where $\mu = y/(2x)^{\frac{1}{2}} > 1$. The first of these scales is of course wider than the second for $\sigma \gg 1$. As θ varies with $\exp(-\eta^2)$ the first region will be an inner layer, the second the outer layer. Recasting (3) in the '*inner' variable* η, and expanding for large values of σ, yields

$$f'(\eta) = \frac{u}{4}\left(\frac{\sigma}{x}\right)^{\frac{1}{2}} = \frac{1}{2}\left[1 - \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}}(\eta + \text{ierfc}(\eta)) + O|\sigma^{-1}|\right], \tag{4}$$

where $f = f(\eta)$, and the prime denotes differentiation with respect to the argument; and

$$\bar{\theta}(\eta) = \theta x^{+\frac{1}{2}} = \exp(-\eta^2). \tag{5}$$

In the '*outer' variable*, however,

$$F'(\mu) = \frac{u}{4}\left(\frac{\sigma}{x}\right)^{\frac{1}{2}} = \frac{\pi^{\frac{1}{2}}}{2}[1 - \sigma^{-1} + O|\sigma^{-2}|].\text{ierfc}(\mu) + \text{exponentially small terms.} \tag{6}$$

Thus, an analytical solution for asymptotically large values of σ is clearly not physically meaningful: we shall proceed through the method of matched asymptotic expansions. Substituting (4) and (5) into (1),

$$\frac{d^3f}{d\eta^3}+\sigma^{-\frac{1}{2}}\bar{\theta}+\frac{2}{\sigma}\left(\eta\frac{d^2f}{d\eta^2}-\frac{df}{d\eta}\right) = 0, \tag{7}$$

$$\frac{d^2\bar{\theta}}{d\eta^2}+2\frac{d}{d\eta}(n\bar{\theta}) = 0. \tag{8}$$

Thus, for $\sigma \to \infty$, the inner layer reduces to $df^3/d\eta^3 \sim 0$; that is, $f' \sim a_0$ when account is taken of the first boundary conditions in (2). In a similar manner, insight into the behaviour of the outer layer may be gained.

3. The case of large Prandtl number

Newtonian fluid of constant physical properties is assumed, incompressible except in as far as the dependence of density upon temperature is concerned, and the Boussinesq approximations are considered valid. Then it is well known that, with certain restrictions, boundary-layer approximations may be used even for the cases of $\sigma \gg 1$ and $\sigma \ll 1$ (Kuiken 1968*a*; Rotem & Claassen 1969*a*). Upon similarity transformations the two-dimensional boundary-layer form of the equations of momentum and energy reduces (following Yih 1956) to

$$\frac{d^3F}{d\zeta^3}+\frac{1}{5}\left[3F\frac{d^2F}{d\zeta^2}-\left(\frac{dF}{d\zeta}\right)^2\right]+H = 0, \tag{9}$$

$$\frac{d^2H}{d\zeta^2}+\frac{3\sigma}{5}\frac{d}{d\zeta}(FH) = 0. \tag{10}$$

Here the reference value adopted to render the velocity dimensionless is $u_c = G^{\frac{1}{2}}\nu/l$, the reference temperature $\dot{q}/K$; the functions F and H are linked to the dimensionless velocity and temperature in (11) and (12) below. ν is the kinematic viscosity of the fluid, l is a characteristic reference length conveniently chosen so as to render the maximum order of the vertical co-ordinate x unity, G is Grashof number ($G \gg 1$) based upon l† and the characteristic reference temperature‡, $\dot{q}$ is the heat dissipation from the source per unit length and time, while K is the thermal conductivity of the fluid. A dimensionless stream function ψ is given by

$$\psi = G^{\frac{1}{5}}x^{\frac{3}{5}}F(\zeta), \tag{11}$$

whereby the function $F(\zeta)$ is defined. The variable ζ is given by

$$\zeta = G^{\frac{1}{5}}y\,x^{-\frac{2}{5}}, \tag{12}$$

where $yG^{\frac{1}{5}}$ is a lateral 'stretched' boundary-layer co-ordinate normal to x (see Rotem & Claassen 1969*b*). The flow configuration is assumed to be infinitely

† As the system is devoid of a characteristic length, the vertical distance from the line source is conveniently chosen to render the co-ordinates dimensionless.

‡ That is, $G = g\beta\dot{q}l^3/(K\nu)$.

wide in the third, z, direction. The dimensionless temperature defines a function H, thus

$$\theta = Cx^{-\frac{3}{5}} H(\zeta), \tag{13}$$

where C is a constant. The boundary conditions, subject to which (9) and (10) have to be solved, are

$$\left.\begin{aligned} F = d^2F/d\zeta^2 = 0, \quad H' = 0 \quad (\zeta = 0), \\ dF/d\zeta = 0, \quad H = 0 \quad (\zeta \to \infty). \end{aligned}\right\} \tag{14}$$

Now, conservation of energy specifies that the total flux in the vertical direction be invariant above the source; therefore,

$$\int_{-\infty}^{\infty} \left(H \frac{dF}{d\zeta}\right) d\zeta = C_1. \tag{15}$$

For convenience the scaling constant C_1 is normalized to unity. Due to the symmetry across a vertical plane passing through the source,

$$\int_0^{\infty} (HF')\, d\zeta = \tfrac{1}{2}.$$

The form of the fundamental solution for which σ may be set to infinity will be determined first. This had already been undertaken by Spalding & Cruddace (1961) who assumed on the basis of physical arguments that the 'inner' region would reduce to a singular sheet of fluid at a velocity varying with x but constant with y. Within this infinitely thin sheet of fluid the dimensionless temperature decreases from its maximum value at any given x to the ambient value (the latter set to zero). The fluid outside this layer is set in motion by viscous shear at the interface rather than by buoyancy. It was then found possible to determine the first-order term of the velocity of the 'inner' region, through an elegant use of an integrated form of the equation of motion. However higher approximations could not be obtained.

The model problem discussed in § 2 above shows the way in which the derivation of the fundamental term can be rendered less intuitive, and in which higher approximations may be derived.

4. The 'inner' solution, $\sigma \gg 1$

Introduce asymptotically stretched co-ordinates and variables as follows:

$$\hat{\zeta} = \sigma^{\frac{1}{2}}\zeta, \quad \hat{H}(\hat{\zeta}) = \sigma^{-\frac{1}{2}} H(\zeta), \quad \hat{F}(\hat{\zeta}) = \sigma^{\frac{1}{2}} F(\zeta). \tag{16}$$

The stretching of ζ to $\hat{\zeta}$ renders the inner layer of width-order unity. Inserting into (9) and (10) gives

$$\frac{d^3\hat{F}}{d\hat{\zeta}^3} + \sigma^{-\frac{1}{2}}\hat{H} + \frac{1}{5\sigma}\left[3\hat{F}\frac{d^2\hat{F}}{d\hat{\zeta}^2} - \left(\frac{d\hat{F}}{d\hat{\zeta}}\right)^2\right] = 0, \tag{17}$$

$$\frac{d^2\hat{H}}{d\hat{\zeta}^2} + \frac{3}{5}\frac{d}{d\hat{\zeta}}(\hat{F}\hat{H}) = 0. \tag{18}$$

Note that here the only term retained in the momentum equation (reduced to 'inner' co-ordinates) represents viscous forces. This is in sharp contrast to the buoyant flow along a heated plate, where the inner equation expresses the equal order of magnitude of the viscous and buoyancy terms. Equations (17) and (18) have the form of (7) and (8). The boundary conditions become

$$\hat{F} = d^2\hat{F}/d\zeta^2 = d\hat{H}/d\zeta = 0, \quad \text{at} \quad \zeta = 0, \tag{19}$$

and (15). The boundary conditions given are not sufficient to determine the solution: we must have for $\zeta \to \infty$ a matching with an 'outer' solution (in the limit as $\sigma \to \infty$ with the passage to the limit *in that order*). The solutions will be assumed *ad hoc* to take the following form:

$$\left.\begin{aligned} \hat{F}(\zeta,\sigma) &= \hat{F}_0(\zeta) + \sigma^{-\frac{1}{2}}\hat{F}_1(\zeta) + \hat{\Phi}(\sigma).\hat{F}_2(\zeta) + \dots, \\ \hat{H}(\zeta,\sigma) &= \hat{H}_0(\zeta) + \sigma^{-\frac{1}{2}}\hat{H}_1(\zeta) + \hat{\phi}(\sigma)\hat{H}_2(\zeta) + \dots. \end{aligned}\right\} \tag{20}$$

The functions $\hat{\Phi}$ and $\hat{\phi}$ cannot be completely determined until both approximations have been calculated. Then they will have to be chosen so as to ensure the proper exponential decay of the temperature for the inner solution, and of the vorticity for the outer.

Clearly the only zeroth-order solution satisfying both (17) and (19) is

$$\hat{F}_0(\zeta) = a_0\zeta, \tag{21}$$

where the constant a_0 will be determined later from the matching conditions. Inserting into (18),

$$\hat{H}_0(\zeta) = \left(\frac{3}{10\pi a_0}\right)^{\frac{1}{2}} \times \exp\left(-\tfrac{3}{10}a_0\zeta^2\right). \tag{22}$$

In (22) the arbitrary scaling constant was chosen so as to satisfy the integral condition (15). The first correction term to both the $\hat{F}$ and $\hat{H}$ profiles is now obtained by inserting the expansions (20) into (17) and (18), equating terms with equal powers of σ, and observing that (15) is already satisfied identically by the fundamental term: for higher-order terms the only requirement is that there be no contribution to the integral in the limit as $\sigma \to \infty$. The first correction terms are

$$\hat{F}_1(\zeta) = \frac{-1}{a_0}\left(\frac{\zeta}{2}\right)^2 + a_1\zeta + \frac{5}{3a_0^2}\, i^2\operatorname{erfc}\left(\zeta\left(\frac{3a_0}{10}\right)^{\frac{1}{2}}\right) - \frac{5}{12a_0^2}, \tag{23}$$

$$\begin{aligned} \hat{H}_1 = -\frac{3}{5}\left(\frac{3}{10\pi a_0}\right)^{\frac{1}{2}} \exp\left(-\tfrac{3}{10}a_0\zeta^2\right) &\left[\frac{5a_1}{6a_0} - \frac{5}{(3a_0)^2} \times \left(\frac{5}{3\pi a_0}\right)^{\frac{1}{2}} - \frac{5\zeta}{3(2a_0)^2}\right. \\ &\left. + \frac{a_1}{2}\zeta^2 - \frac{\zeta^3}{12a_0} - \frac{5}{3a_0^2}\left(\frac{10}{3a_0}\right)^{\frac{1}{2}} \times i^3\operatorname{erfc}\left(\zeta\left(\frac{3a_0}{10}\right)^{\frac{1}{2}}\right)\right]. \end{aligned} \tag{24}$$

The constant a_1 remains again undetermined until matching is carried out. The second-order correction will now be determined: assuming $\Phi(\sigma) = \sigma^{-1}$ gives

$$\frac{d^3\hat{F}_2}{d\zeta^3} = \frac{a_0^2}{5} - \hat{H}_1(\zeta). \tag{25}$$

Using the expression for $\hat{H}_1(\xi)$ given in (24), the integrated form of (25) in the limit as ξ becomes very large is

$$\frac{d^2\hat{F}_2}{d\xi^2} = \frac{a_0^2}{5}\xi + a_2 + \text{transcendentally small terms} \quad (\xi \gg 1). \tag{26}$$

The constant a_2 can consequently be determined:

$$a_2 = \frac{a_1}{2a_0^2} - \frac{5}{6a_0^3}\left(\frac{3}{5\pi a_0}\right)^{\frac{1}{2}}. \tag{27}$$

The second gradient of F_2 has to vanish on the axis of symmetry, and, for very small ξ, the following holds:

$$\left.\frac{d^2\hat{F}_2}{d\xi^2}\right|_{\xi \ll 1} = \left\{\frac{a_0^2}{5} + \frac{1}{a_0\, 2^{\frac{1}{2}}}\left[\frac{a_1}{2}\left(\frac{3}{5\pi a_0}\right)^{\frac{1}{2}} - \frac{1+2^{\frac{1}{2}}/2}{3\pi a_0^2}\right]\right\}.\xi + \text{terms } O|\xi^2|. \tag{28}$$

As further terms of the inner expansion become increasingly complicated, while adding comparatively little physical insight, the expansion will not be continued any further.

5. The 'outer' solution, $\sigma \gg 1$

The outer layer ensures the exponentially rapid decay of the velocity from the maximum at its inner limit to zero at its outer boundary. The temperature has already decreased (to fundamental and first orders) exponentially rapidly in the inner layer: therefore, the fundamental outer temperature term should reduce to the statement $\theta = 0$: the outer layer is that region in which the viscous and inertial terms must balance, buoyancy playing no role, and the velocities in both regions should be of comparable order of magnitude. Thus, the stretching to 'outer' variables involves nothing more than a statement of the vanishing of H in the original equations (9) and (10). For the outer regions, variables superscripted with a tilda will be used. The integral condition (15) is already fulfilled identically by the fundamental term of the inner expansion and applies here no longer. The boundary conditions are, $d\tilde{F}/d\tilde{\xi} \to 0$ as $\tilde{\xi} \to \infty$ and matching with the inner expansion as $\tilde{\xi} \to 0$ ($\sigma \to \infty$) *in that order*. Inserting into (9), (10) tilda-superscripted variables and

$$\tilde{F}(\tilde{\xi}) = \tilde{F}_0(\tilde{\xi}) + \sigma^{-\frac{1}{2}}\tilde{F}_1(\tilde{\xi}) + \tilde{\Phi}(\sigma)\,\tilde{F}_2(\tilde{\xi}) + \dots \tag{29}$$

$\tilde{F}_0$ and $\tilde{F}_1$ are governed by

$$\frac{d^3\tilde{F}_0}{d\tilde{\xi}^3} + \frac{1}{5}\left[3\tilde{F}_0\frac{d^2\tilde{F}_0}{d\tilde{\xi}^2} - \left(\frac{d\tilde{F}_0}{d\tilde{\xi}}\right)^2\right] = 0, \tag{30}$$

$$\frac{d^3\tilde{F}_1}{d\tilde{\xi}^3} + \frac{1}{5}\left[3\tilde{F}_0\frac{d^2\tilde{F}_1}{d\tilde{\xi}^2} - 2\frac{d\tilde{F}_0}{d\tilde{\xi}}\frac{d\tilde{F}_1}{d\tilde{\xi}} + 3\frac{d^2\tilde{F}_0}{d\tilde{\xi}^2}\tilde{F}_1\right] = 0. \tag{31}$$

As expected, only the fundamental term gives rise to a non-linear differential equation. Integrating (17) to order $\sigma^{-\frac{1}{2}}$, and using both (15) and (22), we obtain (following Spalding & Cruddace)

$$\left|\frac{d^2\hat{F}}{d\xi^2} + \frac{\sigma^{-\frac{1}{2}}}{2(d\hat{F}/d\xi)}\right|_{\xi\to\infty} = 0. \tag{32}$$

Using outer variables we therefore have

$$\left.\frac{d^2\tilde{F}}{d\zeta^2}\cdot\frac{d\tilde{F}}{d\zeta}\right|_{\zeta\to 0} = -\tfrac{1}{2}. \tag{33}$$

Finally, the matching condition for the fundamental terms, in inner variables, becomes,

$$\lim_{\substack{(\zeta\to\gg 1)\\(\sigma\to\infty)}} (a_0\zeta\sigma^{-\frac{1}{2}}) \sim \lim_{\substack{(\zeta\to 0)\\(\sigma\to\infty)}} \tilde{F}_0(\zeta). \tag{34}$$

Then the matching condition for (30) is $\tilde{F}_0(0) = 0$. Using the three boundary conditions, $\tilde{F}(0) = \tilde{F}'(\infty) = 0$ and (33), equation (30) is integrated numerically. The results are given in figure 1. The boundary conditions upon the next-order outer approximation are obtained by writing out the matching conditions to that order. For convenience, this was done in inner variables; outer variables throughout would have yielded the equivalent result. The inner expansion is

$$a_0\zeta+\sigma^{-\frac{1}{2}}\times\left[\frac{-1}{a_0}\left(\frac{\zeta}{2}\right)^2+a_1\times\zeta+\frac{5}{3a_0^2}\times i^2\,\mathrm{erfc}\,(\hat{z})-\frac{5}{12a_0^2}\right]$$
$$+\sigma^{-1}\times\left[\frac{a_0^2\zeta^3}{30}+\frac{a_2}{2}\times\zeta^2+\text{small terms}\right]+\ldots. \tag{35}$$

As $\zeta\to\infty$, erfc $(\hat{z})\to 0$; therefore, the inner expansion may be written, for $\zeta\gg 1$,

$$a_0\zeta+\sigma^{-\frac{1}{2}}\times\left[\frac{-1}{a_0}\left(\frac{\zeta}{2}\right)^2+a_1\zeta-\frac{5}{12a_0^2}\right]+\sigma^{-1}$$
$$\times\left[\frac{a_0^2\zeta^3}{30}+\tfrac{1}{2}\times\left(\frac{a_1}{2a_0}-\frac{5}{6a_0^3}\cdot\left(\frac{3}{5\pi a_0}\right)^{\frac{1}{2}}\right)\zeta^2+\text{small terms}\right]+\ldots. \tag{36}$$

The *outer* expansion, written in inner variables, becomes

$$\sigma^{+\frac{1}{2}}\times\hat{F}_0(0)+\left[\zeta\frac{d\hat{F}_0(0)}{d\zeta}+\hat{F}_1(0)\right]+\sigma^{-\frac{1}{2}}\times\left[\frac{\zeta^2}{2!}\times\frac{d^2\hat{F}_0(0)}{d\zeta^2}+\zeta\frac{d\hat{F}_1(0)}{d\zeta}+\hat{F}_2(0)\right]$$
$$+\sigma^{-1}\times\left[\frac{\zeta^3}{3!}\times\frac{d^3\hat{F}_0(0)}{d\zeta^3}+\frac{\zeta^2}{2!}\times\frac{d^2\hat{F}_1(0)}{d\zeta^2}+\zeta\times\frac{d\hat{F}_2(0)}{d\zeta}+\hat{F}_3(0)\right]+\ldots\ (\zeta\ll 1). \tag{37}$$

Some of the boundary conditions are therefore already determined. Further ones are

$$\tilde{F}_0(0) = 0,\quad \frac{d\tilde{F}_0(0)}{d\zeta} = a_0,\quad \frac{d^2\tilde{F}_0(0)}{d\zeta^2} = \frac{-1}{2a_0} \tag{38}$$

(elimination of a_0 between the two last expressions yields again (33)), and

$$\tilde{F}_1(0) = 0,\quad \frac{d\tilde{F}_1(0)}{d\zeta} = a_1,\quad \frac{d^2\tilde{F}_1(0)}{d\zeta^2} = \frac{1}{2a_0}\left\{a_1-\frac{5}{3a_0}\left(\frac{3}{5\pi a_0}\right)^{\frac{1}{2}}\right\}. \tag{39}$$

a_1 may now be obtained from (31) with the aid of (39). The result of the numerical integration gives the following values for the constants a_i: $a_0 = 0{\cdot}93342$, $a_1 = 0{\cdot}32313$, and $a_2 = -0{\cdot}2779$. This compares with a value for a_0 of 0·9335 found by Spalding & Cruddace (1961).

It now remains only to obtain the *uniformly valid approximation*, to the order $\sigma^{-\frac{1}{2}}$ calculated. Using (20), with subscript c for 'composite',

$$F_c'(\xi, \tilde{\zeta}, \sigma) \sim \hat{F}'(\xi) + \tilde{F}'(\tilde{\zeta}) - (\text{the part common to both expansions})$$

$$= \tilde{F}_0'(\tilde{\zeta}) + \sigma^{-\frac{1}{2}} \times \left[\tilde{F}_1'(\tilde{\zeta}) - \frac{5}{3a_0^2}\left(\frac{3a_0}{10}\right)^{\frac{1}{2}} \times \operatorname{ierfc}\left(\left(\frac{3a_0}{10}\right)^{\frac{1}{2}} \xi\right)\right] + O|\sigma^{-1}|, \quad (40)$$

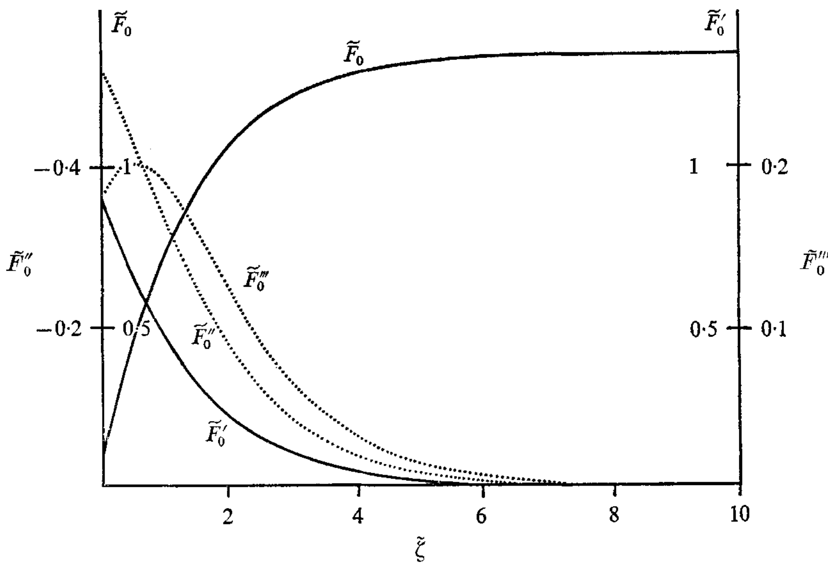

FIGURE 1. Fundamental term, outer solution, $\sigma \gg 1$.

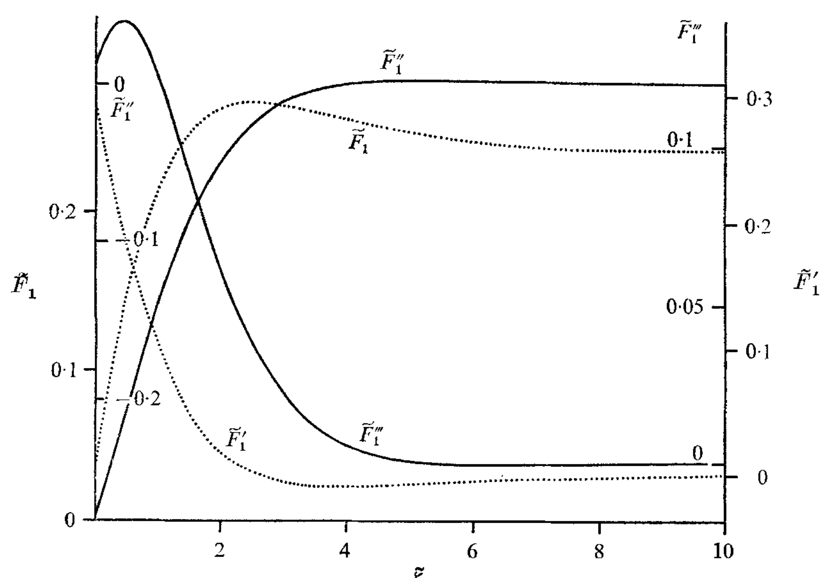

FIGURE 2. First-order correction, outer solution, $\sigma \gg 1$.

$$H_c(\zeta,\tilde{\zeta},\sigma) = \left(\frac{3}{10\pi a_0}\right)^{\frac{1}{2}} \exp\left(-\frac{3a_0}{10}\hat{\zeta}^2\right) \times \left\{1-\tfrac{3}{5}\sigma^{-\frac{1}{2}} \times \left[\frac{5}{6}\frac{a_1}{a_0} - \frac{5}{(3a_0)^2} \times \left(\frac{5}{3\pi a_0}\right)^{\frac{1}{2}} - \frac{5\hat{\zeta}}{3(2a_0)^2} + \frac{a_1}{2}\times\hat{\zeta}^2 - \frac{5}{3a_0^2}\left(\frac{10}{3a_0}\right)^{\frac{1}{2}} \times i^3 \operatorname{erfc}\left(\left(\frac{3a_0}{10}\right)^{\frac{1}{2}}\hat{\zeta}\right) - \hat{\zeta}^3/(12\times a_0)\right]\right\} + O|\sigma^{-1}|. \quad (41)$$

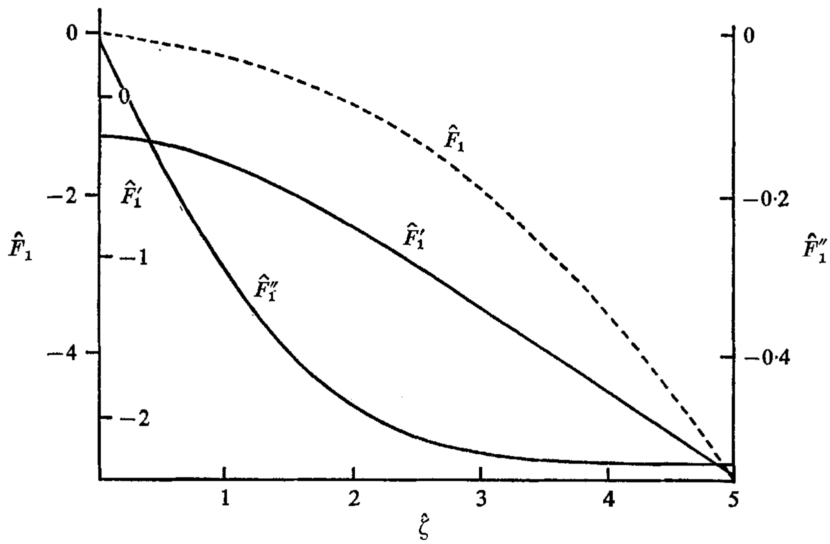

FIGURE 3. First-order correction, inner solution, $\sigma \gg 1$.

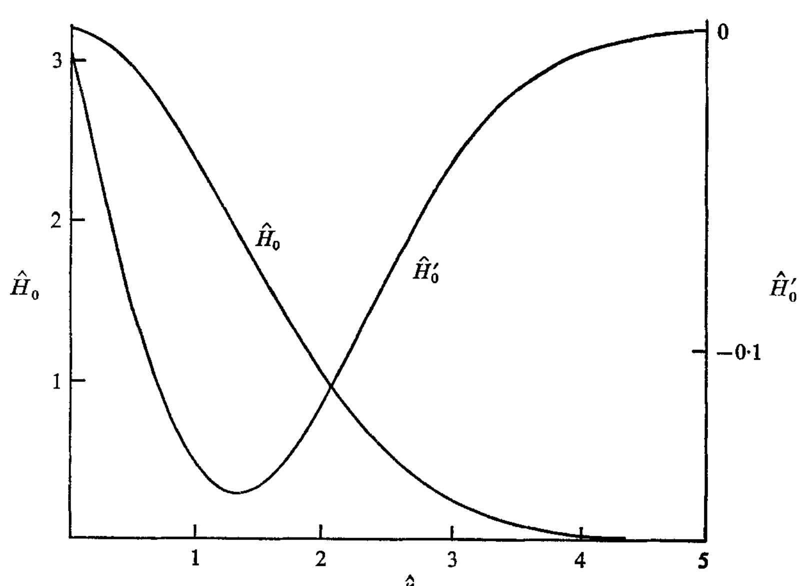

FIGURE 4. Fundamental solution for the temperature function, $\sigma \gg 1$.

For computed results of $\tilde{F}_0, \tilde{F}_0', \tilde{F}_0'', \tilde{F}_0'''$; $\tilde{F}_1, \tilde{F}_1', \tilde{F}_1'', \tilde{F}_1'''$; $\hat{F}_1, \hat{F}_1', \hat{F}_1''$; $\hat{H}_0, \hat{H}_0'$; $\hat{H}_1, \hat{H}_1'$, see the plots of figures 1–5. The composite expansions are shown in figures 6–7. Important numerical results, which cannot be read with sufficient accuracy from the figures, are given in table 1.

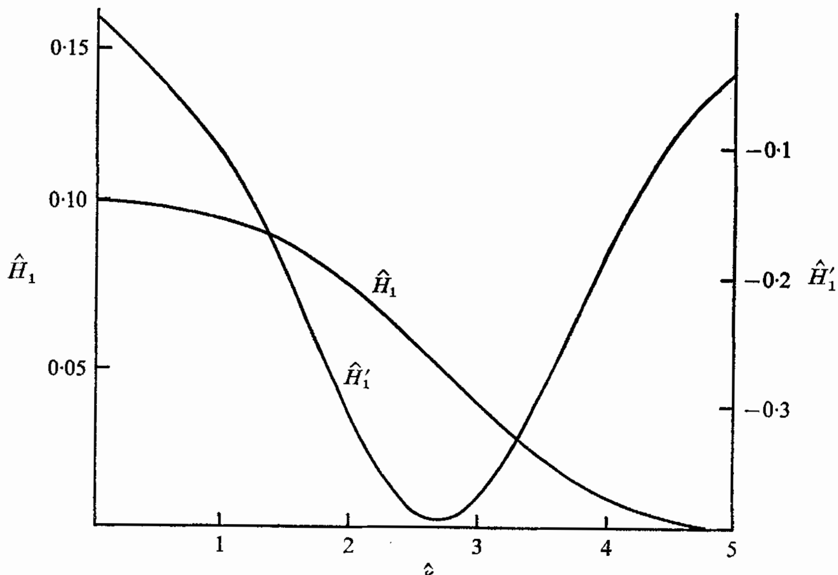

FIGURE 5. First-order correction term to temperature function, $\sigma \gg 1$.

$\sigma \gg 1$

$\tilde{F}_0'(0) = 0{\cdot}9334$	$\tilde{F}_0''(0) = -0{\cdot}5357$	$\tilde{F}_0'''(0) = 0{\cdot}1743$	$\tilde{H}_0(0) = 0$	$\tilde{F}_0(\infty) = 1{\cdot}355$
$\tilde{F}_1'(0) = 0{\cdot}3231$	$\tilde{F}_1''(0) = -0{\cdot}2781$	$\tilde{F}_1'''(0) = 0{\cdot}1206$	$\tilde{H}_1(0) = 0$	$\tilde{F}_1(\infty) = 0{\cdot}2381$
$\hat{F}_0'(0) = 0{\cdot}9334$	$\hat{F}_0''(0) = 0$	$\hat{H}_0(0) = 0{\cdot}31985$	$\hat{H}_0'(0) = 0$	$\tilde{F}_0'(\infty) = 0{\cdot}9334$
$\hat{F}_1'(0) = -0{\cdot}2480$	$\hat{F}_1''(0) = 0$	$\hat{H}_1(0) = 0{\cdot}10212$	$\hat{H}_1'(0) = 0$	

$\sigma \ll 1$

$\tilde{f}_0'(0) = 1{\cdot}50792$		$\tilde{h}_0(0) = 0{\cdot}45603$	$\tilde{h}_0'(0) = 0$	$\tilde{f}_0(\infty) = 1{\cdot}51535$†
$\hat{f}_0'(0) = 1{\cdot}50729$	$\hat{f}_1'(0) = -0{\cdot}84440$			

† For low σ the proper location for infinity in the independent variable $\tilde{\xi}$ is checked by computing satisfactory agreement between

$$\int_0^\infty \tilde{f}'\tilde{f}'\,d\tilde{\xi} \quad \text{and} \quad (\tfrac{5}{4})\int_0^\infty \tilde{h}\,d\tilde{\xi}.$$

TABLE 1

6. Very small values of the Prandtl number

The Prandtl number is a measure of the ratio of the thickness of the momentum boundary layer to that of the thermal boundary layer. Therefore, for the case of very small values of this parameter, the effect of buoyancy extends into the bulk of the fluid surrounding the plume to a distance much larger than the effect of fluid viscosity. Over most of the fluid field the buoyancy and inertia forces will now balance, and the role of viscosity will be confined to a narrow inner layer, where it counteracts fluid inertia and ensures the disappearance of the rate of strain in the plane of symmetry. Therefore, over most of the fluid field, the flow behaves inviscidly to fundamental-term approximation. As the viscous term is in general the one of highest order in the differential equations describing the flow, there will be an inability to fulfill at least one of the boundary conditions imposed physically (the one stipulating the disappearance of the rate-of-strain in the median plane); i.e. an expansion solution will be singular. In the outer layer, both the conductive and convective terms have to be retained in the energy equation.

7. The outer solution, $\sigma \ll 1$

The requisite co-ordinate-stretching transformations take the form,

$$\tilde{f}(\tilde{\xi}) = \sigma^{\frac{2}{5}} F(\zeta), \quad \tilde{h}(\tilde{\xi}) = \sigma^{-\frac{2}{5}} H(\zeta), \quad \tilde{\xi} = \sigma^{\frac{3}{5}}\zeta. \tag{42}$$

These relationships are the equivalents of (16), which held for the case of very large Prandtl number, and the only ones which will retain the integral defined in (15) of finite order. The equations (9), (10), (15) now read:

$$\sigma \frac{d^3\tilde{f}}{d\tilde{\xi}^3} + \frac{1}{5}\left[3\tilde{f}\frac{d^2\tilde{f}}{d\tilde{\xi}^2} - \left(\frac{d\tilde{f}}{d\tilde{\xi}}\right)^2\right] + \tilde{h} = 0, \tag{43}$$

$$\frac{d^2\tilde{h}}{d\tilde{\xi}^2} + \frac{3}{5}\frac{d(\tilde{f}\tilde{h})}{d\tilde{\xi}} = 0, \quad \int_0^\infty \tilde{h}\frac{d\tilde{f}}{d\tilde{\xi}}\,d\tilde{\xi} = \tfrac{1}{2}. \tag{44}$$

The fundamental solution will be obtained by setting $\sigma = 0$ in (43). Equation (44) will be fulfilled by the fundamental solution, the requirement upon the higher-order solutions being that their contribution to the integral vanish.

The boundary conditions become $\tilde{f}_0(0) = \tilde{f}_0'(\infty) = \tilde{h}_0(\infty) = 0$. An interpretation of the first condition, which is obtained from matching with the inner solution, is that in outer variables the inner layer forms a singular sheet just as in the case of large Prandtl number. Inspection of (43) for $\sigma = 0$ shows that it has a non-removable singularity at $\tilde{\xi} = 0$: this need not complicate numerical computation greatly (Rotem & Claassen 1969*b*). In the present context, an expansion in powers of $\tilde{\xi}$ (not necessarily integral) is required (Kuiken 1969). The expansion to fundamental order is

$$\tilde{f}_0(\tilde{\xi}) = b_1\tilde{\xi} + b_2\tilde{\xi}^{\frac{5}{3}} - \frac{5}{42}\frac{b_2^2}{b_1}\tilde{\xi}^{\frac{7}{3}} + O|\tilde{\xi}^3|, \tag{45}$$

$$\tilde{h}_0(\tilde{\xi}) = \tfrac{1}{5}b_1^2 - \tfrac{3}{50}b_1^3\tilde{\xi}^2 - \tfrac{9}{200}b_1^2 b_2\tilde{\xi}^{\frac{8}{3}} + \ldots, \tag{46}$$

where the constants b_1 and b_2 are obtained from numerical integration:

$$b_1 = 1{\cdot}50729 \quad \text{and} \quad b_2 = -0{\cdot}61085.$$

The fundamental outer solutions and their derivatives have also been calculated (figure 8).† The *inner* solution, through the influence of the viscous terms, causes the rate of shear on the median plane to vanish. Consequently, this is not a requirement which the outer expansion has to fulfill: in outer variables the inner solution acts like a stretchable drag plate.

8. The inner solution, $\sigma \ll 1$

In the narrow inner region the influence of viscosity forces is such as to satisfy the conditions of analyticity and symmetry of the velocity profile near the median plane. The condition that the first gradient of the *temperature* vanish is already satisfied by the outer solution. Therefore, the inner layer will not influence the lateral heat dissipation from the plume to first order, resembling closely the cases of the heat transfer from plates at asymptotically small Prandtl number. The velocities in both inner and outer layers have to be of the same order of magnitude, while the contribution of the inner layer to the flux integral should vanish. All these requirements are fulfilled by the transformations,

$$\hat{f}(\hat{\xi}) = \sigma^{-\frac{1}{10}}F(\zeta), \quad \hat{h}(\hat{\xi}) = \sigma^{-\frac{2}{5}}H(\zeta), \quad \hat{\xi} = \sigma^{+\frac{1}{10}}\zeta. \tag{47}$$

It is apparent that these transformations lead to a scaling-up of f (the dimensionless stream-function equivalent), and to a stretching to order unity of the width of the layer, which will also be of constant temperature to first order. The requisite equations are

$$\frac{d^3\hat{f}}{d\hat{\xi}^3}+\frac{1}{5}\left[3\hat{f}\frac{d^2\hat{f}}{d\hat{\xi}^2}-\left(\frac{d\hat{f}}{d\hat{\xi}}\right)^2\right]+\hat{h} = 0, \tag{48}$$

$$\frac{d^2\hat{h}}{d\hat{\xi}^2}+\frac{3}{5}\sigma\frac{d}{d\hat{\xi}}(\hat{f}\hat{h}) = 0, \tag{49}$$

and $\hat{f}(0) = \hat{f}''(0) = \hat{h}'(0) = 0$. The fundamental-order solution is obtained by setting σ to zero in these equations. The supplementary matching conditions, to that order, stipulate

$$[\lim_{\hat{\xi}\to\infty}\hat{f}(\hat{\xi}) \sim b_1\hat{\xi}, \quad \lim \hat{h}(\hat{\xi})_{\hat{\xi}\to\infty} = b_1^2/5]_{(\sigma\to 0)}.$$

The higher-order matching is now carried out in the standard way. It is convenient to express the outer solution in inner variables, and pass to the limit $\tilde{\xi} \to 0$, $\sigma \to 0$ in that order, whence the behaviour of the *outer* expansions,

$$\hat{f}(\hat{\xi}) \sim b_1\hat{\xi}+\sigma^{\frac{1}{3}}b_2\hat{\xi}^{\frac{5}{3}}+\text{terms of } O|\sigma^{\frac{2}{3}}|, \tag{50}$$

$$\hat{h}(\hat{\xi}) \sim \frac{b_1^2}{5}-\tfrac{3}{50}\sigma b_1^3\hat{\xi}^2+\text{higher-order terms.} \tag{51}$$

The next term in the inner expansion, $\sigma^{\frac{1}{3}}\hat{f}_1(\hat{\xi})$ should satisfy

$$\frac{d^3\hat{f}_1}{d\hat{\xi}^3}+\frac{b_1}{5}\left[3\hat{\xi}\frac{d^2\hat{f}_1}{d\hat{\xi}^2}-2\frac{d\hat{f}_1}{d\hat{\xi}}\right] = 0. \tag{52}$$

Therefore,
$$\hat{f}_1(\hat{\xi}) = b_2\left(\frac{10}{3b_1}\right)^{\frac{1}{3}}\frac{5\Gamma(\frac{5}{6})}{3\pi^{\frac{1}{2}}}\hat{\xi}\times M\left(-\tfrac{1}{3},\tfrac{3}{2},-\frac{3b_1}{10}\hat{\xi}^2\right), \tag{53}$$

† Further results may be obtained from Z.R.

where $M(a, b, z)$ is Kummer's function. Equation (53) gives the expected asymptotic behaviour for large values of $\hat{\xi}$.

Higher approximations may now be calculated, and have to be computed numerically to give uniformly valid solutions to the order of approximation dealt with.† Their addition is hardly justified by added physical information. The composite velocity becomes

$$f_c'(\tilde{\xi}, \sigma, \hat{\xi}) = f_0'(\tilde{\xi}) + \sigma^{\frac{1}{3}} \left[b_2 \left(\frac{10}{3b_1} \right)^{\frac{1}{3}} \frac{5\Gamma(\frac{5}{6})}{3\pi^{\frac{1}{2}}} \left[M\left(-\tfrac{1}{3}, \tfrac{3}{2}, -\frac{3b_1}{10}\hat{\xi}^2 \right) + \tfrac{2}{15} b_1 \hat{\xi}^2 M\left(\tfrac{2}{3}, \tfrac{5}{2}, -\frac{3b_1}{10}\hat{\xi}^2 \right) \right] - \tfrac{5}{3} b_2 \hat{\xi}^{\frac{2}{3}} \right]. \quad (54)$$

The composite solution is traced graphically in figure 9, together with the exact solutions obtained through direct numerical computation for $\sigma = 0{\cdot}01$ and $0{\cdot}1$.

9. Results and discussion

This work supplements the information available on the behaviour of two-dimensional, laminar, buoyant plumes, by considering the two asymptotic cases of infinitely large and vanishingly small values of the Prandtl modulus. The solutions present some features not in general found in this type of analysis, which are due to the boundary conditions that obtain. 'Universal' solutions to the flow configuration considered are derived, in the sense that the dependence upon the exact value of the Prandtl number is reduced to a simple multiplicative effect.

Figures 6, 7 and 9 compare the asymptotic approximations obtained with data calculated directly numerically. Whereas in the case of natural convection from solid boundaries a value of $\sigma = 5$ comes very close to the asymptotically large σ case, while for $0{\cdot}1$ the same is true for the other extreme, this does not hold as well in the buoyant plume configuration. Moreover, as discussed in Rotem & Claassen (1969*a*), a more rapid approximation in the temperature profiles than in those of the velocity distribution is notable.

It is remarkable that buoyancy forces appear to have no direct influence upon the zeroth-order solution for the case of $\sigma \to \infty$. Thus, the only reason for which a plume can arise at very high values of σ is to maintain energy conservation, which necessitates the removal of the source flux by a vertical velocity. The mechanism of the *inception* of the plume cannot be given unless the near vicinity of the source be considered separately. As far as the present work is concerned, the assumption is that the only singularity arises due to the parameter σ. Now, an underlying assumption for the validity of the boundary-layer equations, assumed from the outset, is that $(G^{\frac{1}{4}} \times x^{\frac{3}{4}})$ be a large value. Here the Grashof number G may be based on any convenient reference length which renders the co-ordinate x of order unity, while this latter is measured from the source of heat in a direction opposite to that of the vector gravity. Thus, the system considered here does not possess a reference length *in the large*; and this is in fact the prerequisite for the possibility of similarity solutions.

† The computed values are available from Z. R.

However, when the region close to the source is considered, it is found that there is indeed a characteristic length in the small. Or, differently put, no matter how large the Grashof number, there will always be a range sufficiently near to the source where the present analysis will break down: in that region viscous

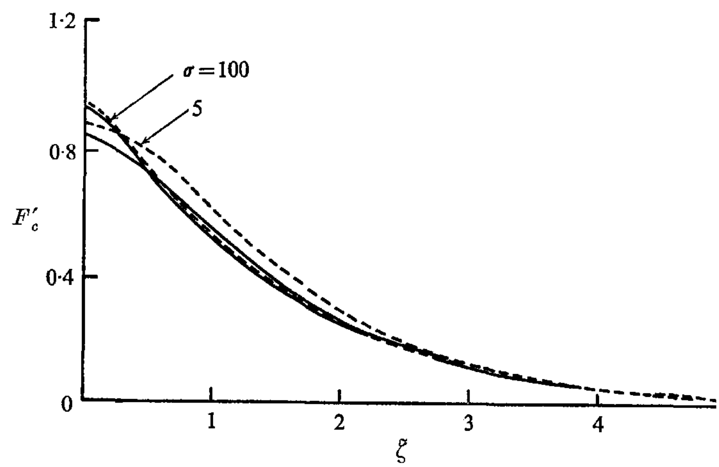

FIGURE 6. Composite velocity function, $\sigma \gg 1$: ——, composite solution; . . . . ., exact.

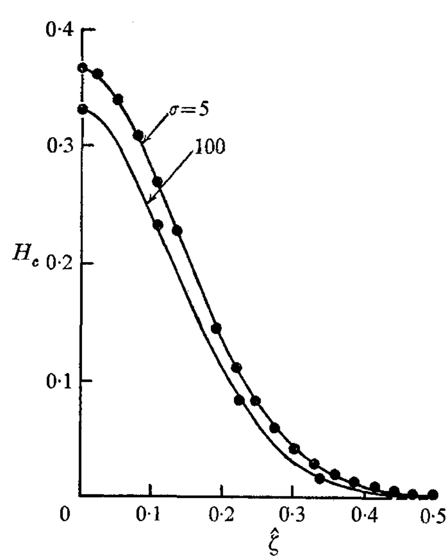

FIGURE 7. Composite temperature function, $\sigma \gg 1$: ——, composite solution; . . . ., exact.

and buoyancy forces will predominate. The existence of this inner region has recently received attention by Mahony (1956), Fendell (1968) and Hieber & Gebhart (1969).

Some note should be taken of the fact that for the outer solution of the vanishingly small Prandtl number case, vorticity does not seem to decay exponentially rapidly with the lateral co-ordinate approaching the matching point with the outer flow. Two complementary physically plausible explanations are available to explain this effect. (i) In the first instance, it should be realized that the matching is not with an outer *potential* flow field; thus, some of the arguments as to

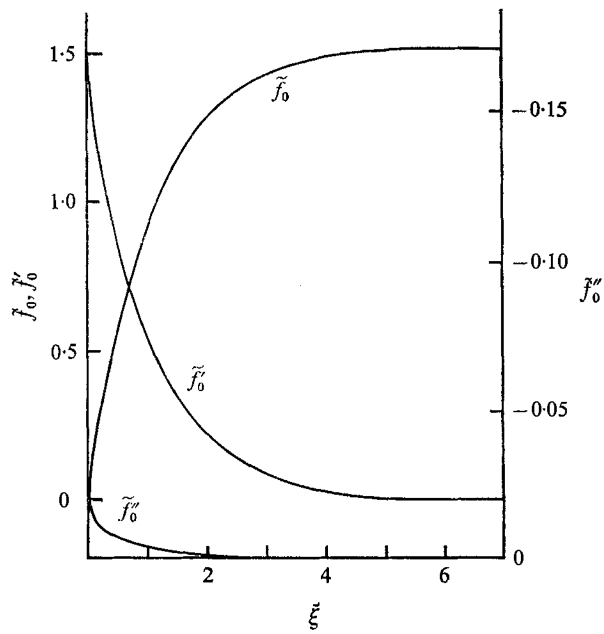

FIGURE 8. Fundamental term, outer solution, $\sigma \ll 1$.

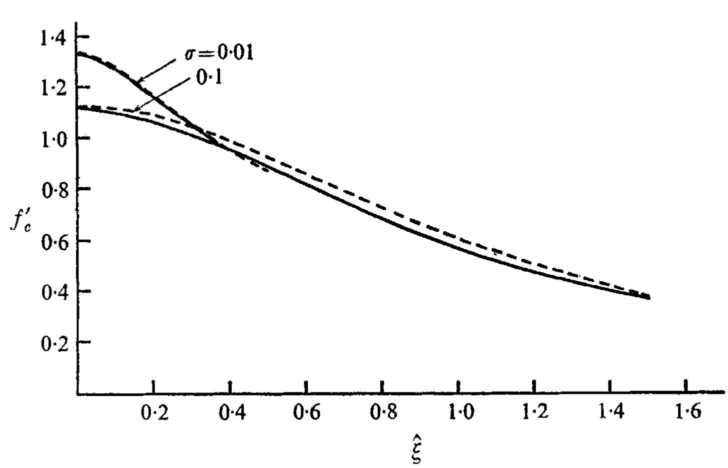

FIGURE 9. Composite velocity function, $\sigma \ll 1$: ——, composite solution; . . . . ., exact.

the necessity of exponential decay hardly apply to the present situation. (ii) The limit $\sigma \to 0$ may be interpreted as the increase in the thermal conductivity of the buoyant fluid beyond all limits. This immediately results in an infinitely fast conductive release of heat from the line source, effecting in a similar manner the buoyancy of the outer fluid. Problems such as these have been discussed in a different and wider context by Brown & Stewartson (1965). Those authors and also Rotem (1966) discuss other examples of related effects.

The authors are grateful for the financial support and encouragement of the Canada National Research Council.

REFERENCES

Brown, S. N. & Stewartson, K. 1965 *J. Fluid Mech.* **23**, 673.

Cole, J. D. 1968 *Perturbation Methods in Applied Mathematics.* Blaisdell.

Fendell, F. E. 1968 *J. Fluid Mech.* **33**, 163.

Hieber, C. A. & Gebhart, B. 1969 *J. Fluid Mech.* **38**, 137.

Kuiken, H. K. 1968*a* *J. Eng. Math.* **2**, 95.

Kuiken, H. K. 1968*b* *J. Eng. Math.* **2**, 355.

Kuiken, H. K. 1969 *J. Fluid Mech.* **37**, 785.

Mahony, J. J. 1956 *Proc. Roy. Soc.* A **238**, 412.

Rotem, Z. 1966 *Chem Engng Sci.* **21**, 618.

Rotem, Z. & Claassen, L. 1969*a* *J. Fluid Mech.* **38**, 173.

Rotem, Z. & Claassen, L. 1969*b* *Can. J. Chem. Engng* **47**, 561.

Spalding, D. B. & Cruddace, R. G. 1961 *Int. J. Heat Mass Transfer* **3**, 55.

Yih, C. S. 1956 *Proceedings of the First Symposium on the Use of Models in Geophysics*, Washington, 117.

Zeldovich, Y. B. 1937 *Zhur. Exp. Teor. Fiz.* **7**, 1463.